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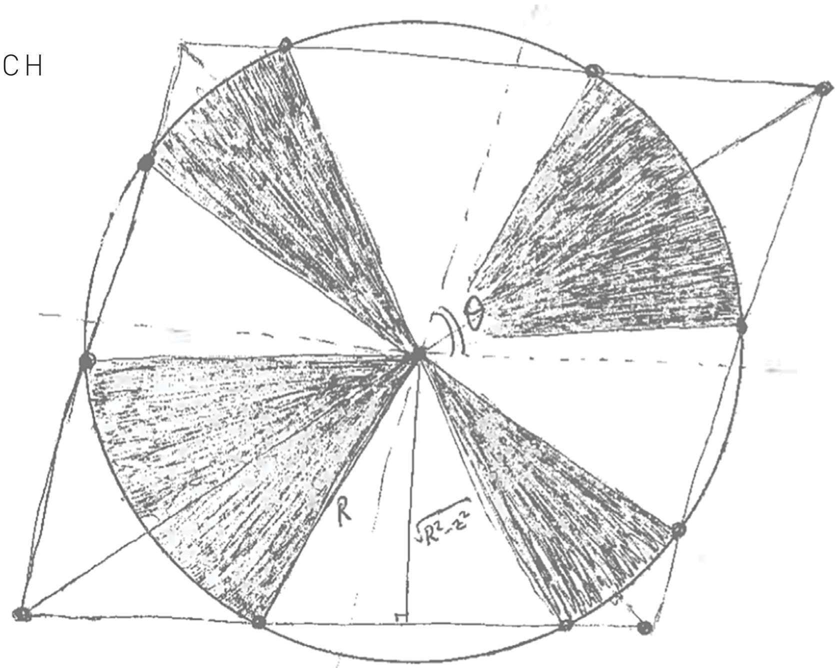


The Mouhefanggai and Cavalieri's Principle

BY JARED LICHTMAN

The Mouhefanggai

The "Mouhefanggai", dating back to Ancient China, is the region enclosed by two cylinders with equal radii that meet perpendicularly, as shown in figure 1. More precisely, given $R > 0$ we find the volume of the following region in $\mathbb{R}^3$

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \leq R^2, y^2 + z^2 \leq R^2\}$$

For each $z \in [-R, R]$, the cross section of S at a fixed height z is given by

$$S(z) = \{(x, y, z) \in \mathbb{R}^3 : |x|, |y| \leq \sqrt{R^2 - z^2}\}$$

This describes a square of side length $2\sqrt{R^2 - z^2}$, as shown in figure 2. Then the volume of S is given by

$$V(S) = \int_{-R}^{R} A(S(z))\, dz = \int_{-R}^{R} (2\sqrt{R^2 - z^2})^2\, dz$$
$$= 8\int_{0}^{R} R^2 - z^2\, dz = \frac{16}{3}R^3$$

We see that the computation was carried out by summing the intersections at each cross-section, effectively reducing computing volumes to computing areas, which is much easier. This method is known as Cavalieri's principle, and we will use it to obtain similar results that would be quite diffcult any other way.

While we're here, we can obtain some other properties of the Mouhefanggai. For example, since its edges contribute zero surface area, we may compute its surface area as the derivative of its volume, with respect to R. That is, $A = dV/dR = 16R^2$. Additionally, the Mouhefanggai has 4 congruent faces, 4 edges, and 2 vertices, giving an Euler characteristic $\chi = 2$ in agreement with that of the sphere. Thus the surface area of each face is $4R^2$.

Generalizing the Cross-Section

The configuration of cylinders used to make the Mouhefanggai may be generalized in myriad ways. For example, instead of intersecting perpendicularly, the two cylinders might meet at a given angle $\theta \in (0, \pi/2]$. We will apply the same technique used above but we now stray from the formalism, seeing as the

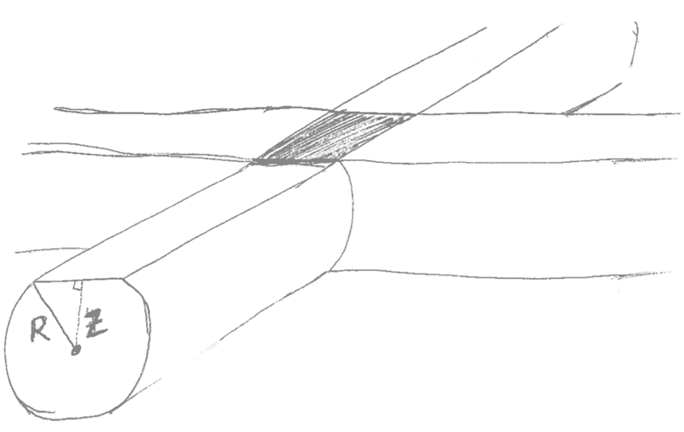


Figure 1: Cross-section of Mouhefanggai at height *z*.

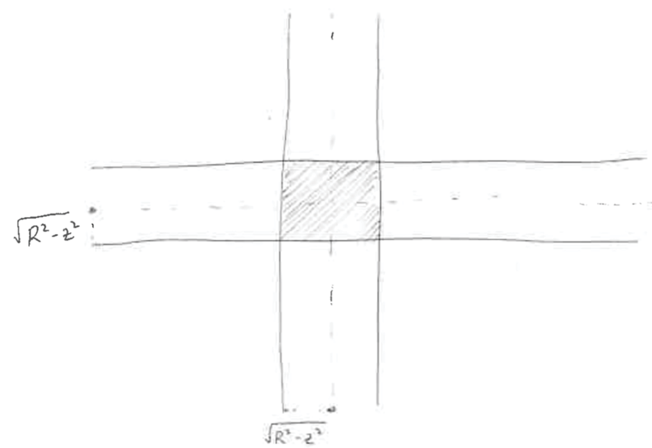


Figure 2: Bird's eye view of cross-section at height *z*.

problem definition is clear. For each $z \in [-R, R]$, the cross section at a fixed height z are given by two infinite rectangles of width $2(R^2 - z^2)$ whose midlines meet at an angle θ. As shown in figure 3, the region of intersection is a rhombus, which, by elementary geometry, has area $4(R^2 - z^2) \csc \theta$. Then the volume is given by

$$V = \int_{-R}^{R} 4(R^2 - z^2) \csc\theta \, dz = \frac{16}{3} R^3 \csc\theta$$

Observe that this result differs from the original problem by a factor of $\csc \theta$. In the case when $\theta = \pi/2$ we recover the original volume $16R^3$. Also note that as θ tends to 0, the volume tends to infinity. This agrees with our intuition, seeing as when $\theta = 0$, the two cylinders are parallel—actually coincident—so their intersection is the entire infinite cylinder. As before, there are 4 congruent faces, 4 edges, and 2 vertices, with Euler characteristic $\chi = 2$. And the total surface area is $A = dV/dR = 16R^2 \csc \theta$, so the area of each face is $4R^2 \csc \theta$.

Another path of generalization is to introduce more cylinders. For example, suppose n evenly spaced cylinders intersect. That is, there are n cylinders of radius R whose axes are all coplanar, and meet at a point such that adjacent axes meet at an angle $\varphi = 2\pi/n$. We proceed in the same fashion, considering the cross section at a fixed height z. As shown in figure 4, the intersection is a regular $2n$-gon whose apothem is $R^2 - z^2$. The area is then $2n(R^2 - z^2)\tan(\pi/2n)$, so the volume of intersection is

$$V = \int_{-R}^{R} 2n(R^2 - z^2) \tan\varphi dz =$$

$$= 4n \tan\varphi \int_{0}^{R} R^2 - z^2 dz = \frac{8n}{3} R^3 \tan\varphi$$

When $n = 2$ we recover the Mouhefanggai volume of $(16/3)R^3$. Also, observe that since $\lim_{x\to 0}(\tan x)/x = 1$, we have

$$\lim_{n\to\infty} V = \lim_{n\to\infty} \frac{8n}{3} R^3 \tan\varphi$$

$$= \lim_{n\to\infty} \frac{4\pi}{3} R^3 \left(\frac{\tan\varphi}{\varphi}\right) = \frac{4\pi}{3} R^3$$

which is the volume of the insphere of radius R. Here there are $2n$ congruent faces, $2n$ edges, and 2 vertices, giving an Euler characteristic $\chi = 2$. The total surface area is $A = dV/dR = 8nR^2\tan\varphi$, and so each face has area $4R^2\tan\varphi$.

Cylinder Axes in Three Dimensions

The next problem forces us to return to three dimensions, at least initially. Consider three cylinders whose axes are pairwise orthogonal and intersect at a point (i.e. the x, y, and z axes).

As depicted in figures 5-8, the cross-sections are naturally split up in two, one for cross- sections at heights $0 < |z| < R/\sqrt{2}$ and the other at heights $R/\sqrt{2} < |z| < R$. For the former, as shown in figure 6, the the square of intersection of the first two cylinders intersects partially with the third cylinder. This results in four triangular regions and four sectors. Using elementary geometry, the area of the cross-section is $4z\sqrt{R^2 - z^2} + (\pi - 4\sin^{-1}((z/R))R^2$. For the latter, as shown in figure 8, the square of intersection of the first two cylinders is contained entirely in the third cylinder, so we may proceed identically as done for the Mouhefanggai. Thus the volume of intersection is

$$V = 2\int_{0}^{R/\sqrt{2}} 4z\sqrt{R^2 - z^2}$$

$$+ \left(\pi - 4\sin^{-1}(z/R)\right) R^2 \, dz + 2\int_{R/\sqrt{2}}^{R} (2\sqrt{R^2 - z^2})^2 \, dz$$

$$V = \int_{0}^{R/\sqrt{2}} 8z\sqrt{R^2 - z^2} + 2\pi R^2 - 8\sin^{-1}(z/R) R^2 \, dz$$

$$+ 8\int_{R/\sqrt{2}}^{R} (R^2 - z^2) \, dz$$

$$V = \Big[-\frac{8}{3}(R^2 - z^2)^{3/2} + 2\pi R^2 z$$

$$- 8R^2(\sqrt{R^2 - z^2} + z\sin^{-1}(z/R)) \Big]_{0}^{R/\sqrt{2}}$$

$$+ 8\big[R^2 z - z^3/3\big]_{R/\sqrt{2}}^{R}$$

$$V = \frac{2}{3}R^3(16 - 7\sqrt{2}) + \frac{2}{3}R^3(8 - 5\sqrt{2}) = 8(2 - \sqrt{2})R^3$$

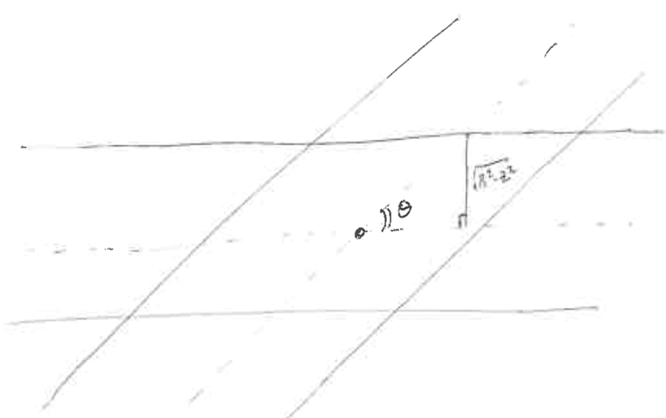


Figure 3: Cross-section of two cylinders meeting at angle θ. The area of intersection rhombus is (base)×(height) = $(2\sqrt{(R^2 - z^2)}) \times (2\sqrt{(R^2 - z^2)} \csc\theta)$ = $4(R^2 - z^2)\csc\theta$.

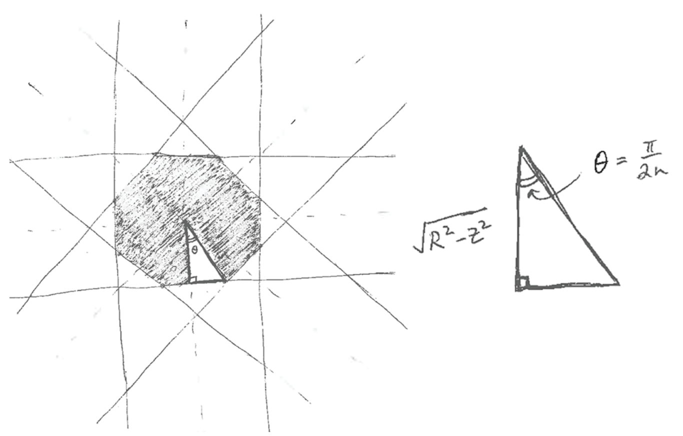
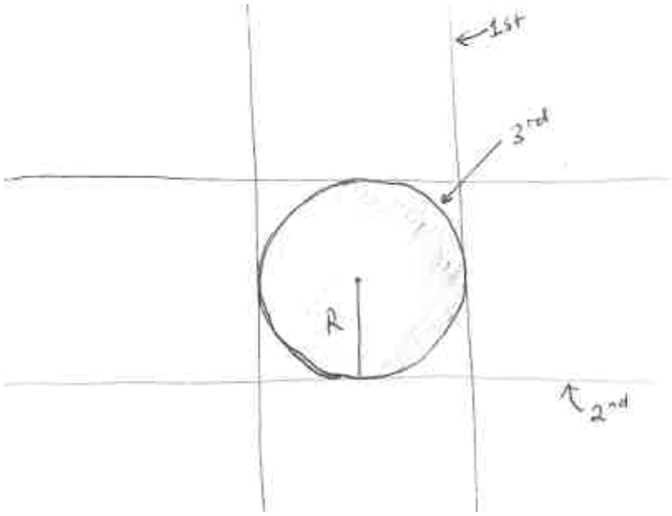
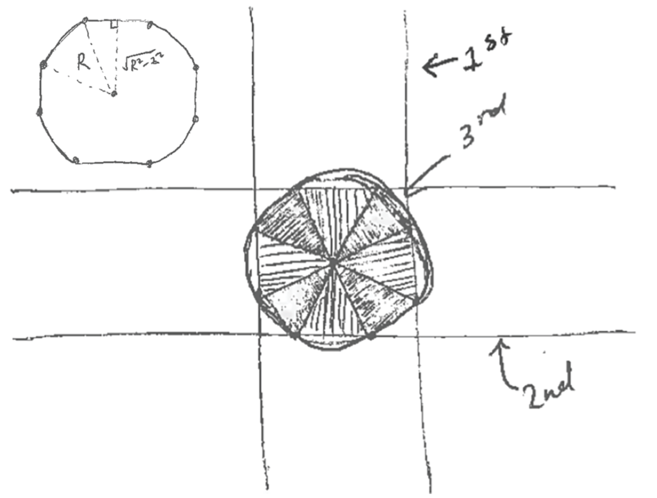
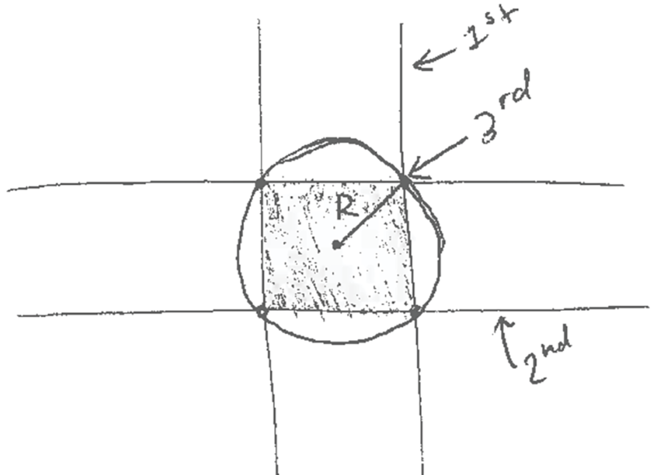


Figure 4: Cross-section of n cylinders whose intersection is a regular $2n$-gon.







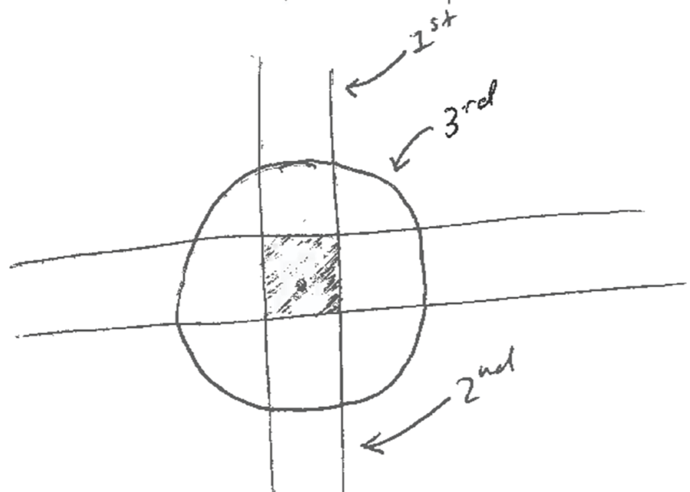


Figure 5 (top left): Cross-section of three mutually orthogonal cylinders at height $z = 0$. **Figure 6 (top right):** Cross-section of three cylinders at height $0 < |z| < R/\sqrt{2}$. The area of intersection is given by $4(z\sqrt{(R^2 - z^2)}) + (\pi R^2)((2\pi - 8\sin^{-1}(z/R)/(2\pi)) = 4z\sqrt{(R^2 - z^2)} + R^2(\pi - 4\sin^{-1}(z/R)$. **Figure 7 (bottom left):** Cross-section of three mutually orthogonal cylinders at height $|z| = R/\sqrt{2}$. **Figure 8 (bottom right):** Cross-section of three cylinders at height $R/\sqrt{2} < |z| \leqslant R$. Here the intersection is contained entirely in cylinder 3, so proceed as in Mouhefanggai.

Alternatively, we may exploit the symmetry of the configuration. As shown in figure 7, at the critical height $|z| = R/\sqrt{2}$, the square of side length $R\sqrt{2}$ fits entirely into the third cylinder. Permuting the three cylinders, we see that in fact the *cube* of side length $R\sqrt{2}$ sits in the volume of intersection (and maximally so, i.e. the cube is circumscribed). Therefore we may take the volume of this cube, and add to it six times (one per cube face) the volume of a "cap" (the region in the latter case from above, where the shape is unchanged from the Mouhefanggai between heights $z = R/\sqrt{2}$ and R). Thus the volume of intersection is

$$V = (\sqrt{2}R)^3 + 6\int_{R/\sqrt{2}}^{R} (2\sqrt{R^2 - z^2})^2\, dz$$

$$= 2\sqrt{2}R^3 + 24\int_{R/\sqrt{2}}^{R} (R^2 - z^2)\, dz$$

$$= 2\sqrt{2}R^3 + 2R^3(8 - 5\sqrt{2}) = 8(2 - \sqrt{2})R^3$$

Based on the observation of the inscribed cube, we find that there are 12 faces (one per cube edge), 24 edges (four per cube face) and 14 vertices (one per cube face plus original cube vertices, i.e. two "types" of vertices). Therefore the resulting net corresponds to the rhombic-dodecahedron, which is the dual of the cube-octahedron. The Euler characteristic is again $\chi = 2$. The total surface area is $A = dV/dR = 24(2 - \sqrt{2})R^2$, and so each face has area $2(2 - \sqrt{2})R^2$.

We may now revisit previous problems and add a cylinder along the z-axis. First, consider n evenly spaced coplanar cylinders with another cylinder along the z-axis mutually perpendicular to the others. As with the three mutually orthogonal cylinders, the cross-sections break down into two cases. The $2n$-gon is either completely contained in the cylinder, or partially intersects. Letting $\varphi = \pi/2n$, this transition occurs at $|z| = R\sin\varphi$ when the cylinder circumscribes the $2n$-gon, seen in figure 9. Thus when $|z| > R\sin\varphi$ we may treat the problem as before, ignoring the last cylinder. And when $|z| < R\sin\varphi$, as shown in figure 10, the region of intersection is made up of $2n$ congruent triangles (each with area $z\sqrt{R^2 - z^2}$ and $2n$ sectors that sweep a total angle of $2\pi - 4n\sin^{-1}(z/R)$. Thus the volume of intersection is

$$V = 2\int_0^{R\sin\varphi} 2nz\sqrt{R^2 - z^2}$$

$$+ \Big(\pi - 2n\sin^{-1}(z/R)\Big)R^2\, dz$$

$$+ 2\int_{R\sin\varphi}^{R} 2n(R^2 - z^2)\tan\varphi\, dz$$

$$= \int_0^{R\sin\varphi} 4nz\sqrt{R^2 - z^2} + 2\pi R^2$$

$$- 4nR^2 \sin^{-1}(z/R)\, dz$$

$$+ 4n\tan\varphi \int_{R\sin\varphi}^{R} (R^2 - z^2)\, dz$$

$$= \Big[-\frac{4n}{3}(R^2 - z^2)^{3/2} + 2\pi R^2 z$$

$$- 4nR^2(\sqrt{R^2 - z^2} + z\sin^{-1}(z/R)) \Big]_0^{R\sin\varphi}$$

$$+4n\tan\varphi \big[R^2 z - z^3/3\big]_{R\sin\varphi}^{R}$$

$$= \frac{4n}{3}R^3\Big(4 - 3\cos\varphi - \cos^3\varphi$$

$$+ \tan\varphi\big(2 - 3\sin\varphi + \sin^3\varphi\big)\Big)$$

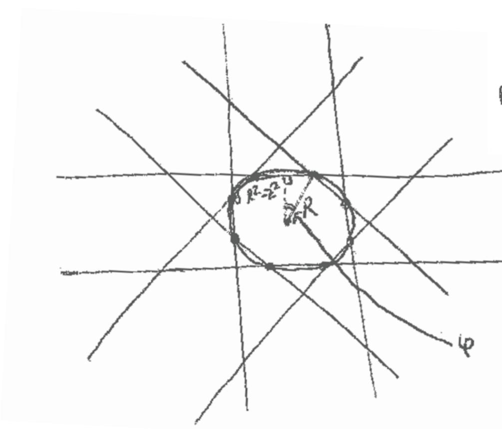
Note that when $n = 2$, $\varphi = \pi/4$, we recover the volume $V = 8(2 - \sqrt{2})R^3$ of the three mutually orthogonal cylinders. And

$$\lim_{n\to\infty} V = \lim_{n\to\infty} \frac{4n}{3}R^3\Big(4 - 3\cos\varphi - \cos^3\varphi$$

$$+ \tan\varphi\big(2 - 3\sin\varphi + \sin^3\varphi\big)\Big)$$

$$= \lim_{n\to\infty} \frac{2\pi}{3}R^3\Big(\frac{\tan\varphi}{\varphi}\Big)\big(2 - 3\sin\varphi + \sin^3\varphi\big) = \frac{4\pi}{3}R^3$$

recovering the volume of the insphere.



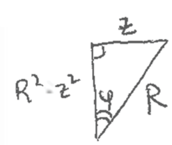
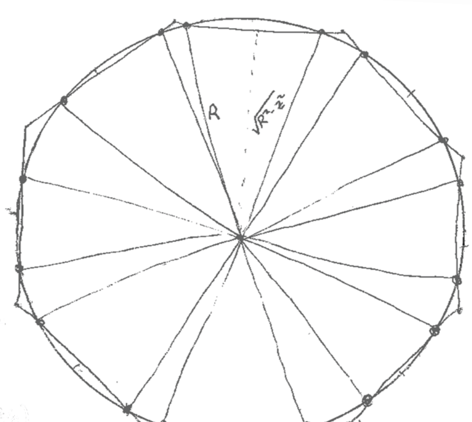


Figure 9: Cross-section of n cylinders with one mutually orthogonal cylinder at the critical height $|z| = R\sin(\varphi)$ where the orthogonal cylinder circumscribes the $2n$-gon cross-section.



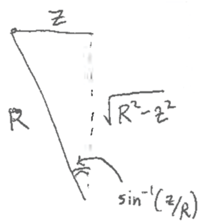


Figure 10: Cross-section of n co-planar cylinders with one mutually orthogonal cylinder at height $|z| < R\sin(\varphi)$.

Finally, consider two cylinders meeting at angle θ with a third mutually orthogonal cylinder. As shown in figure 11, the cross-sections break down into three cases, with transitions occurring at $|z| = R\sin(\theta/2)$ and $R\cos(\theta/2)$. The rhombus has either four, two, or zero corners protruding, where the last means the rhombus is completely contained in the third cylinder. Seen in figure 12, when $0 < |z| < R\sin(\theta/2)$ the region of intersection is made up of four congruent triangles (each with area $z\sqrt{R^2 - z^2}$) and four sectors that sweep a total angle of $2\pi - 8\sin^{-1}(z/R) = 8\cos^{-1}(z/R) - 2\pi$. Then, as shown in figure 13, when $R\sin(\theta/2) < |z| < R\cos(\theta/2)$ the region of intersection is made up of a partial rhombus and two small sectors. And when $R\cos(\theta/2) < |z| < R$, the rhombus is entirely contained, so we may proceed as before. Thus the volume of intersection is

$$V = 2\int_0^{R\sin(\theta/2)} 4z\sqrt{R^2 - z^2}$$

$$+ [4\cos^{-1}(z/R) - \pi]R^2\, dz$$

$$+2\int_{R\sin(\theta/2)}^{R\cos(\theta/2)} 2z\sqrt{R^2 - z^2}$$

$$+ 2(R^2 - z^2)(\csc\theta - \cot\theta)$$

$$+ [2\cos^{-1}(z/R) - \theta]R^2\, dz$$

$$+2\int_{R\cos(\theta/2)}^{R} 4(R^2 - z^2)\csc\theta\, dz$$

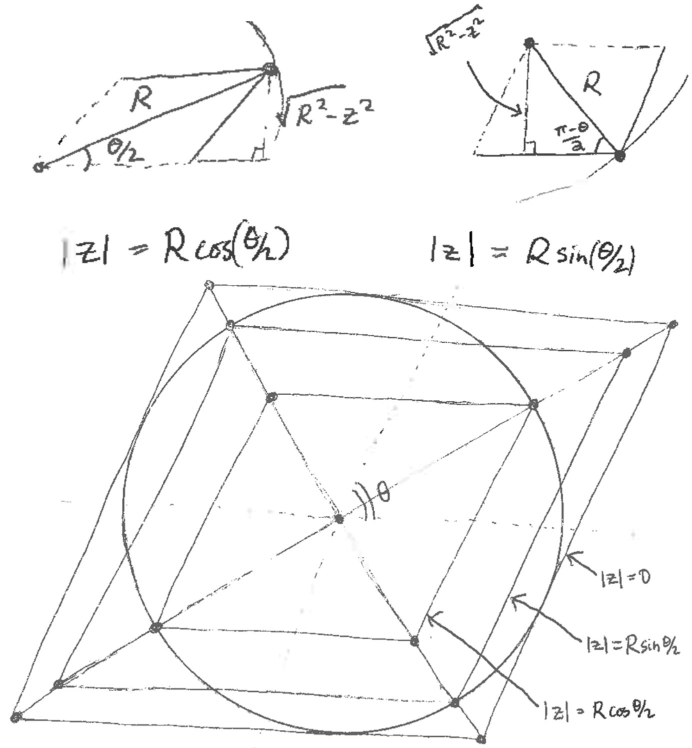


Figure 11: Depiction of critical cross-sections of two cylinders meeting at angle theta with a third mutually orthogonal cylinder. The critical heights are $|z| = R\sin(\theta/2)$, $R\cos(\theta/2)$.

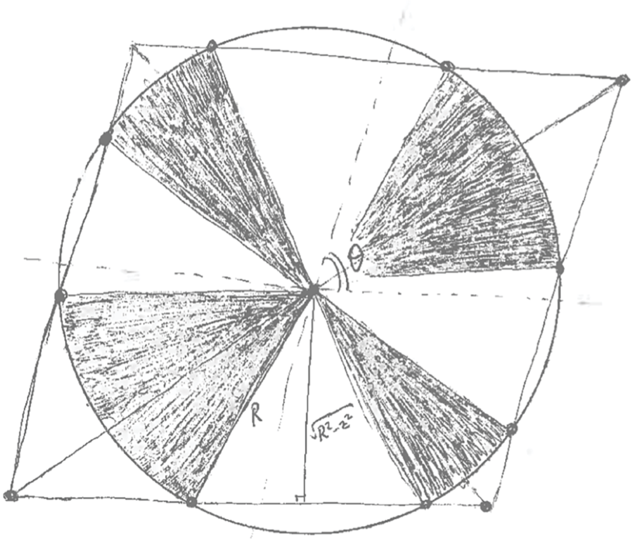


Figure 12: Cross-section when 0 < |z| < Rsin(θ/2) of two cylinders meeting at angle theta with a third mutual orthogonal cylinder.

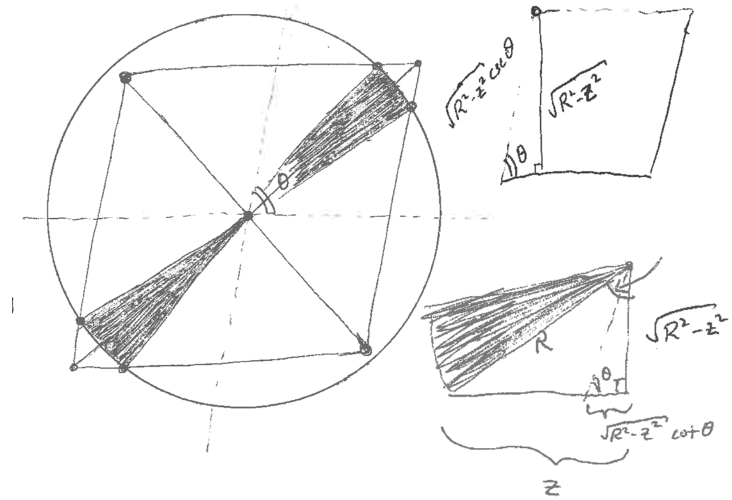


Figure 13: Cross-section when Rsin(θ/2) < |z| < Rcos(θ/2) of two cylinders meeting at angle theta with a third mutual orthogonal. The area of the partial rhombus is $2(R^2 - z^2)\csc\theta + z\sqrt{(R^2 - z^2)} - (R^2 - z^2)\cot\theta$. The angle swept by the sectors is $4\cos^{-1}(z/R) - 2\theta$.

$$
\begin{aligned}
&= 2\Big[-\frac{4}{3}(R^2-z^2)^{3/2} + 4R^2\big[z\cos^{-1}(z/R) \\
&\qquad - \sqrt{R^2-z^2}\big] - \pi R^2 z\Big]_0^{R\sin(\theta/2)} \\
&\quad +2\Big[-\frac{2}{3}(R^2-z^2)^{3/2} \\
&\quad +2(R^2z-z^3/3)(\csc\theta-\cot\theta) \\
&\quad +2R^2\big[z\cos^{-1}(z/R) \\
&\quad -\sqrt{R^2-z^2}\big]-\theta R^2 z\Big]_{R\sin(\theta/2)}^{R\cos(\theta/2)} \\
&\quad +8\csc\theta\Big[R^2z-\frac{1}{3}z^3\Big]_{R\cos(\theta/2)}^{R} \\
&= \frac{8R^3}{3}\Big[4-3\cos(\theta/2)-\cos^3(\theta/2)\Big] \\
&\quad + 2(\pi-2\theta)R^3\sin(\theta/2) \\
&\quad +\frac{2R^3}{3}\big(\cos(\theta/2)-\sin(\theta/2)\big)\big[7+\cos\theta \\
&\quad + 4(\csc\theta-\cot\theta)\big] - 2(\pi-2\theta)R^3\sin(\theta/2) \\
&\quad +\frac{8R^3}{3}\csc\theta\big[2-3\cos(\theta/2)+\cos^3(\theta/2)\big]
\end{aligned}
$$

Hence we have that

$$
\begin{aligned}
V &= \frac{8R^3}{3}\Big[4-3\cos(\theta/2)-\cos^3(\theta/2) \\
&\quad + \csc\theta\big[2-3\cos(\theta/2)+\cos^3(\theta/2)\big]\Big] \\
&\quad +\frac{2R^3}{3}\big(\cos(\theta/2)-\sin(\theta/2)\big)\big[7+\cos\theta \\
&\quad + 4(\csc\theta-\cot\theta)\big]
\end{aligned}
$$

Note that when $\theta = 0$, we recover the volume V = $(16/3)R^3$ of the Mouhefanggai. And when $\theta = \pi/2$, we recover the volume $V = 8(2 - \sqrt{2})R^3$ of the three mutually orthogonal cylinders. It is also worth noting that there is a single extremal angle on the interval (0, π/2), seen in figure 14. Holding R constant, V achieves a local minimum of $V \approx 4.5578R^3$ for $\theta \approx 1.07991$ rad, an angle slightly larger than $\pi/3 \approx 1.04719$. **D**

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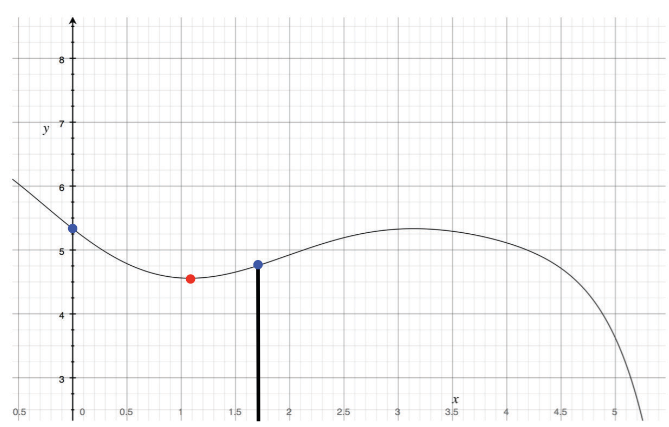

Figure 14: Graph showing extremal angle at θ ≈ 1.07991 rad.

Source: All drawings by Jared Lichtman. Graph generated by Jared Lichtman.